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Quasiperiodic solutions of discretized Korteweg–de Vries equations

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The periodic-type solutions of the semidiscrete (ordinary differential) and difference-difference (functional with shifted arguments) versions of the Korteweg–de Vries equation are considered. Applying the formalism of dispersion equations, the quasiperiodic solutions and solutions in the form of solitons on the background of periodic wave trains are found and discussed.

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Very soon after the discovery of soliton processes for numerous partial differential equations (PDE), the equations discretized with respect to a single independent variable were considered, leading to ordinary differential equations (ODE). They also exhibited the multisoliton and quasiperiodic solutions and the best known are related to the Toda chain.

Recently a growing interest can be observed in the analysis of a certain class of nonlinear functional equations, whose solutions have soliton-type properties, e.g., [1,2]. This relates to the existence of the multisoliton and/or quasiperiodic solutions, together with an application of the modification of inverse scattering method (IST) in the first case, and the Riemann surface philosophy in the second one.

The present paper can be considered as the generalization of results announced in [2], periodic (one-gap) solutions of semidiscrete and difference-difference Korteweg–de Vries (KdV) equations (sd-KdV and dd-KdV, respectively) have been found. In a natural way the first one, sd-KdV, represents the ordinary differential equation, but dd-KdV makes a pure functional equation.

We preserve the terminology of semidiscrete and difference-difference Korteweg–de Vries equation used in [2], after Hirota [11]. It should be stressed, however, that in both cases the independent variables x and t can be considered always as continuous ones. In this context dd-KdV belongs to the class of functional equations with shifted arguments rather than to the class of difference-difference equations. Similarly, sd-KdV represents an ODE also with shifted arguments.

Applying the technique of dispersion equations developed earlier for PDE's, below we present the multiperiodic (multigap) solutions, not referring them, however, to the structure of Riemann surfaces following from IST. As for PDE's, the method is similar to the famous Hirota technique and the most surprising conclusion is that for functional equations this technique also holds.

In this sense the presented paper makes an alternative approach or illustration of the results reported in [1]. Moreover, our formalism allows one to construct the solutions of sd-KdV, dd-KdV, and similar equations in the form of multisolitons on the background of multiphase quasiperiodic solution. An elementary example of a single soliton on the background of periodic solution for dd-KdV concludes this part.

In the last part, a definition of a discrete version of the Hirota differential operator D is given, which, in conjunction with the addition property (1), justifies an application of the Hirota-like formalism to ODE and even to the functional equations.

Let us consider the class of functions $F : C^g \to C$ having the following factorization property:

$$F(z+w)F(z-w) = \sum_{\epsilon} W_\epsilon(w) Z_\epsilon(z), \qquad (1)$$

where the sum is over a finite set (here always Z_2^g) and the functions Z_ϵ and W_ϵ also map $C^g \to$C. It is obvious that if Z_ϵ are linearly independent, W_ϵ is symmetric, i.e., $W_\epsilon(w) = W_\epsilon(-w)$.

Furthermore, if a Riemann Θ function is chosen as the function F, Eq. (1) is known as the addition property of Riemann Θ functions and a few possible realizations of (1) exist.

Let B be a Riemann matrix (i.e., $B \in C^{g\times g}$ is symmetric and with a positively definite imaginary part), $x \in C^g$. Then the Riemann Θ function defined as

$$\Theta(z|B) := \sum_{n\in Z^g} \exp[i\pi[2\langle z,n\rangle + \langle n, Bn\rangle)], \qquad (2)$$

with

$$\langle z,n\rangle := \sum_{i=1}^{g} z_i n_i, \qquad (3)$$

constitutes a particular case of so-called Riemann Θ functions with characteristics $\alpha, \beta \in R^g$, defined as

$$\begin{aligned}\Theta[\alpha,\beta](z|B) := \sum_{n\in Z^g} &\exp\{i\pi[2\langle(z+\beta),(n+\alpha)\rangle \\ &+\langle(n+\alpha),B(n+\alpha)\rangle]\} \\ = &\exp\{\ i\pi[2\langle(z+\beta),\alpha\rangle + \langle\alpha,B\alpha\rangle]\} \\ &\times\Theta(z+\beta+B\alpha|B). \qquad (4)\end{aligned}$$

Of course, if n and m are integer ($m,n \in Z^g$), then

$$\Theta[0,n](z|B) = \Theta(z|B), \qquad (5)$$

and also

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$$\Theta(z+n+Bm|B) = \exp[-i\pi(2\langle z,m\rangle + \langle m,Bm\rangle)]\Theta(z|B). \quad (6)$$

In the case when $x \in C$ and $B \in C$, the multiple sums in (2) and (4) reduce to a single one and then to the Riemann functions of a single variable, $\Theta[0,0](z|B)$, $\Theta[1/2,0](z|B)$, $\Theta[0,1/2](z|B)$, and $\Theta[1/2,1/2](z|B)$, also known as the four basic Jacobi functions. As was mentioned before, for the Riemann Θ functions, there exist a few versions of Eq. (1), depending on how the functions W and Z are interpreted.

Let F functions be the Riemann Θ functions

$$F(z+w) = \Theta(z+w|B) \text{ and } F(z-w) = \Theta(z-w|B), \quad (7)$$

then, either

$$Z_\epsilon(z) = \Theta[\epsilon_1/2, \epsilon_2/2](x|B)^2,$$

$$W_\epsilon(w) = 2^{-g}\Theta[\epsilon_1/2, \epsilon_2/2](x|B)^2/\Theta(0|B)^2, \quad (8)$$

$$\text{with } \epsilon := \epsilon_1, \epsilon_2; \epsilon_1 \in Z_2^g, \epsilon_2 \in Z_2^g,$$

or

$$Z_\epsilon(z) = \Theta(z+\epsilon/2|B)^2,$$

$$W_\epsilon(w) = 2^{-g}\sum_{\mu \in Z_2^g}(-1)^{\langle\epsilon,\mu\rangle}\exp(i2\pi\langle w,\epsilon\rangle) \times\Theta(2w+B\epsilon|2B)/\Theta(B\epsilon|2B), \quad (9)$$

$$\epsilon \in Z_2^g,$$

or

$$Z_\epsilon(z) = \Theta[\epsilon/2,0](2z|2B) = \exp[i\pi(2\langle z,\epsilon\rangle + \langle\epsilon,B\epsilon\rangle/2)]\Theta(2z+B\epsilon|2B),$$

$$W_\epsilon(w) = \Theta[\epsilon/2,0](2w|2B) = \exp[i\pi(2\langle w,\epsilon\rangle + \langle\epsilon,B\epsilon\rangle/2)]\Theta(2w+B\epsilon|2B), \quad \epsilon \in Z_2^g, \quad (10)$$

or

$$Z_\epsilon(z) = \Theta(z+\epsilon/2|B/2), \quad W_\epsilon(w) = 2^{-g}\Theta(w+\epsilon/2|B/2), \quad \epsilon \in Z_2^g. \quad (11)$$

The above equations are useful in the derivation of a system of dispersion equations for some nonlinear partial differential equations of soliton type and also for their discrete or functional variants.

Equations (8) and (10) are the special case of more general statements concerning the addition of Θ functions [3–5]. Relation (9) is particularly convenient if a process in the form of a soliton on the quasiperiodic background is considered [6,7]. On the other hand, a relatively simple formula (11), see [7], can be applied, unfortunately, only to the quasiperiodic solutions, since in the so-called soliton limit some elements of the sum on the right hand side of (1) become singular.

The fundamental consequence of the property (1) is that for the function F one can easily calculate the subsequent derivatives and, what is most important, the basis is always the same.

For example, if the property (1) holds, we have for the first derivatives of the shifted F functions,

$$[\ln[F(z+w)/F(z-w)]]_{,j} = \sum_\epsilon W_{\epsilon,j}(w)Z_\epsilon(z)/[F(z+w)F(z-w)], \quad (12)$$

using a shorthand notation $\partial/\partial z_j := (\cdot)_{,j}$. Denoting $L := \ln F(z)$, for even derivatives we have

$$L(z)_{,ij} = \tfrac{1}{2}\sum_\epsilon W_{\epsilon,ij}(0)Z_\epsilon(z)/F(z)^2, \quad (13)$$

$$L_{,ijkl} + 2(L_{,ij}L_{,kl} + L_{,ik}L_{,jl} + L_{,il}L_{,jk}) = \sum_\epsilon W_{\epsilon,ijkl}(0)Z_\epsilon(z)/F(z)^2, \quad (14)$$

$$L_{,ijklmn} + 2\underbrace{(L_{,ij}L_{,klmn}+\cdots)}_{15\text{ permutations}} + 4\underbrace{(L_{,ij}L_{,kl}L_{,mn}+\cdots)}_{15\text{ permutations}} = \sum_\epsilon W_{\epsilon,ijklmn}(0)Z_\epsilon(z)/F(z)^2, \quad (15)$$

etc. [7].

Note that in (13)–(15) the set of functions of argument z on the right hand side, which forms a basis, is always the same. Moreover, the differential properties (13)–(15) of F functions satisfying (1) determine a certain, previously known, so-called second hierarchy of KdV equations. The first member of this hierarchy, related to (14), is the KdV equation and the second one, related to (15), is the Kotera-Sawada equation.

This fact allows one to write easily the dispersion equation for the above mentioned partial differential equations [7,8]. Here we confine ourselves to the KdV equation, but the same procedure can be applied to other commonly known soliton-type equations.

Starting from the KdV equation in the form $u_t + 6uu_x + u_{xxx} = 0$, and using the substitution $u(x,t) = 2(\ln F)_{xx}$, where $F = F(\kappa x + \omega t)$ and $\kappa, \omega \in C^g$, we arrive at the equation

$$\sum_{i,j=1}^{g}\omega_i\kappa_j L_{,ij} + \sum_{i,j,k,l=1}^{g}\kappa_i\kappa_j\kappa_k\kappa_l \times[L_{,ijkl} + 2(L_{,ij}L_{,kl} + L_{,ik}L_{,jl} + L_{,il}L_{,jk})] = c, \quad (16)$$

where c is an arbitrary constant.

Considering $Z_\epsilon(z)$ as linearly independent, due to (14) Eq. (16) reduces to the system of algebraic equations

$$\sum_{i,j=1}^{g} \omega_i \kappa_j W_{\epsilon,ij}(0) + \sum_{i,j,k,l=1}^{g} \kappa_i\kappa_j\kappa_k\kappa_l W_{\epsilon,ijkl}(0) - cW_\epsilon(0) = 0, \quad (17)$$

for any ϵ specified by (1).

Since (17) involves the propagation vectors κ_j and angular frequencies ω_j, it is natural to call this system of equations a system of dispersion equations (SDE). It is a system of 2^g equations for $2g$ unknown quantities $\kappa_j/c^{1/4}, \omega_j/c^{3/4} (j = 1, \dots g)$ and thus for $g > 2$ it is overdetermined.

Hence for $g > 2$ the SDE supplies the conditions for the elements of the B matrix being a parameter of the solution. In the language of algebraic geometry this is equivalent to the B matrix being a period matrix of a suitable Riemann surface.

We do not intend to discuss this problem further and the reader is referred to the extensive literature on the subject [3,4,9,10].

For $g = 1$, all functions appearing in (1)–(17) are functions of only one variable (Jacobi θ functions) and the SDE reduces then to the simple formula

$$\omega = \frac{-\kappa^3[W_1(0)W_0^{(4)}(0) - W_0(0)W_1^{(4)}(0)]}{[W_1(0)W_0^{(2)}(0) - W_0(0)W_1^{(2)}(0)]}. \quad (18)$$

For $g = 2$, the relations are slightly more complicated. However, for $g = 3$ we get also two conditions for the six elements of the B matrix. It means that among six elements of the B matrix only four can be assumed arbitrarily, in contrast to the case of $g < 3$, when an arbitrary Riemann matrix B is admissible. As a rule, for $g = 2$ Riemann Θ functions cannot be represented by Jacobi (i.e., one-dimensional) functions. There are only a few rather exceptional situations when such expansion is possible.

Now, let us consider the semidiscrete KdV equation [11,2],

$$u_{n,t} = (1 + u_n)^2(u_{n-1/2} - u_{n+1/2}),$$
$$u_n := f_{n-1/2}f_{n+1/2}/f_n^2 - 1. \quad (19)$$

Denoting

$$f_n := F(\kappa n + \omega t), \quad (20)$$

where $\kappa \in C^g$, $\omega \in C^g$ and n is integer, we obtain

$$\sum_{i=1}^{g} \omega_i \left\{ \ln\left[\frac{f_{n+1/4+1/4}}{f_{n+1/4-1/4}}\frac{f_{n-1/4-1/4}}{f_{n-1/4+1/4}}\right]\right\}_{,z_i} = \frac{f_{n-1/4+3/4}f_{n-1/4-3/4}}{f_{n-1/4+1/4}f_{n-1/4-1/4}} - \frac{f_{n+1/4+3/4}f_{n+1/4-3/4}}{f_{n+1/4+1/4}f_{n+1/4-1/4}}. \quad (21)$$

Since F has the property (1), through (12), similarly as in the case of a doubly differential, traditional KdV equation, we obtain

$$\sum_{i=1}^{g} \omega_i W_{\epsilon,i}(\kappa/4) + C_0 W_\epsilon(\kappa/4) + W_\epsilon(3\kappa/4) = 0, \quad (22)$$

for any ϵ for which (1) holds. Here $W_{\epsilon,i}(\kappa/4) := \left.\frac{\partial W_\epsilon}{\partial w_i}\right|_{w=\kappa/4}$, and C_0 is a certain constant, which eventually can also be determined, although it is not necessary. In particular cases W_ϵ can be given by (8)–(11).

Now the conditions of solvability look slightly different than for (17). Although κ appears as an argument of the W function, one can argue that up to $g = 2$ the solution always exists and for $g > 2$ some additional conditions for the parameter (B matrix, if F is interpreted in terms of Θ functions) of the W function have to be satisfied. For $g = 1$, as previously, we have

$$\omega = \frac{-[W_0(3\lambda)W_1(\lambda) - W_1(3\lambda)W_0(\lambda)]}{[W_0'(\lambda)W_1(\lambda) - W_1'(\lambda)W_0(\lambda)]}, \quad (23)$$

where $\lambda = \kappa/4$, and W' denotes the derivative with respect to argument.

For $g = 2$, in the case of quasiperiodic processes, W functions are represented by the Θ functions parametrized by Riemannian matrix B, and the solution exists for any choice of Riemannian matrix $B, (B_{11}, B_{22}, B_{12}, B_{21})$.

If the representation of W functions in terms of the Jacobi θ functions is adopted ($g = 1$) according, e.g., to the scheme (8), the relation (23) can be simplified to

$$\omega = -\Theta[1,1](\kappa|B)/\Theta'[1,1](0|B), \quad (24)$$

and then to the soliton dispersion relation ($B = ib, b \to \infty$)

$$i\pi\omega = -\sinh(i\pi\kappa), \quad (25)$$

when the definition (2) of $\Theta(z|B)$ holds. This equation is identical with that of Ref. [2], if we disregard the $i\pi$ coefficient following from a different definition of Θ functions.

Now let us consider the the functional variant of the KdV equation, denoted in [11,2] as the difference-difference KdV equation,

$$-[1/u_n(t + \delta/2) - 1/u_n(t - \delta/2)]/\delta = u_{n-1/2}(t) - u_{n+1/2}(t), \quad (26)$$

where on the left-hand side we have the discrete version of a derivative with respect to t. The variables n and t can be continuous or discrete and it is convenient to substitute

$$u_n := f_{n-1/2}(t)f_{n+1/2}(t)/f_n(t + \delta/2)f_n(t - \delta/2) - 1. \quad (27)$$

Assuming the solution in the form of (20), introducing a simpler notation $g(z) = F(\kappa n + \omega t + z)$, where $(\kappa n + \omega t + z) \in C^g$, with $z \in C^g$, and denoting also

$$G(\omega\delta,\kappa) := -[g(\omega\delta)g(-\omega\delta/2-\kappa/2)/\delta + g(-\kappa)g(\omega\delta/2+\kappa/2)]/g(\omega\delta/2-\kappa/2)g(0), \tag{28}$$

we obtain from (26)

$$G(\omega\delta,\kappa) = G(-\omega\delta,-\kappa). \tag{29}$$

Since this equation should hold for any n and t, one can look for a solution of (26) assuming that

$$G(\omega\delta,\kappa) = C(\omega\delta,\kappa), \tag{30}$$

where C does not depend on n,t but it is a function of $\omega\delta$ and κ. This leads, of course, to

$$g(\omega\delta)g(-\omega\delta/2-\kappa/2)/\delta + g(-\kappa)g(\omega\delta/2+\kappa/2) + Cg(\omega\delta/2-\kappa/2)g(0) = 0. \tag{31}$$

The definition of g and property (1) of F now yield

$$\sum_\epsilon [W_\epsilon(^1w)/\delta + W_\epsilon(^2w) + CW_\epsilon(^3w)] \times Z_\epsilon(\kappa n + \omega t - {}^3w) = 0, \tag{32}$$

where

$$^1w : = (3\omega\delta+\kappa)/4,\ ^2w := (\omega\delta+3\kappa)/4,$$
$$^3w : = (\omega\delta-\kappa a)/4,\ ^1w,^2w,^3w \in C^g. \tag{33}$$

The solution of the system (32) reduces to the solution of the system of algebraic equations (i.e., dispersion relations)

$$W_\epsilon(^1w)/\delta + W_\epsilon(^2w) + CW_\epsilon(^3w) = 0, \tag{34}$$

where C is again a constant with respect to n and t, dependent on $\delta\omega$ and κ. The set of solutions of dispersion equations (34) contains all the solutions of (32) whenever Z_ϵ form a set of linearly independent functions. [More accurately, there are two branches of solutions determined by equations of the type given by (34), but each branch can be obtained from the other by the transformation $\delta \to -\delta$.]

All remarks concerning the relation (22) can also be applied here. Similarly as before, for fixed g (number of zones) we have a system of 2^g equations (for any $\epsilon \in Z_2^g$) involving $2g+1$ parameters to be determined ($\kappa_i, \omega_i, i = 1,\ldots,g$ and C). For $g = 1$ the dispersion relation reduces to

$$\delta = \frac{-[W_0(^1w)W_1(^3w) - W_1(^1w)W_0(^3w)]}{[W_0(^2w)W_1(^3w) - W_1(^2w)W_0(^3w)]}. \tag{35}$$

If F and consequently Z_ϵ, W_ϵ are interpreted in terms of Riemann Θ functions, solvability of (34) determines the existence of a multigap solution of the dd-KdV equation.

One can ask whether there exists a correspondence between (34) and (22) in the limiting case when δ tends to zero. Let us observe first of all that in (34), close to the point $\delta = 0$, C has to have an evaluation $C = -1/\delta + C_0 + C_1\delta + \cdots$. Substituting C and Taylor expansion of W into (34) and keeping only the terms of order of δ^0, we obtain just Eq. (22).

Similarly, as previously in the context of (23), for periodic processes and $g = 1$, Eq. (35) for periodic solutions can be simplified to the form reported in [2],

$$\Theta[1,1](\pi\omega\delta|B) = \delta\,\Theta[1,1](\pi\kappa|B), \tag{36}$$

and next to the dispersion equation for a single soliton solution

$$\sin(\pi\omega\delta) + \delta\sin(\pi\kappa) = 0. \tag{37}$$

For real δ Eq. (37) has an imaginary solution ($\kappa,\omega \in iR$) — then (27) represents the soliton solution. There also exists a real solution ($\kappa,\omega \in R$) but then (27) represents the singular wave train solution. Examples of both types of solutions are presented in Figs. 1 and 2.

As was mentioned before, different interpretations of functions F, W, and Z appearing in Eq. (1) are possible. The corresponding solutions are of course equivalent up to a modular transformation of Θ functions. If we confine ourselves to the Θ functions, we have at least four possibilities, (8)–(11), to determine W and Z. It is natural to treat the soliton processes as a limiting form of quasiperiodic ones. The details of such a procedure can be found in [7,8]. Let us denote such a limiting process as S-lim (soliton limit), for the time being not defining it precisely.

In order to satisfy relations analogous to (12)–(15), the operation S-lim should have such a property that if (1) holds then also

$$S\text{-lim}[F(z+w)]S\text{-lim}[F(z-w)] = \sum_\epsilon S\text{-lim}[W_\epsilon(w)]S\text{-lim}[Z_\epsilon(z)]. \tag{38}$$

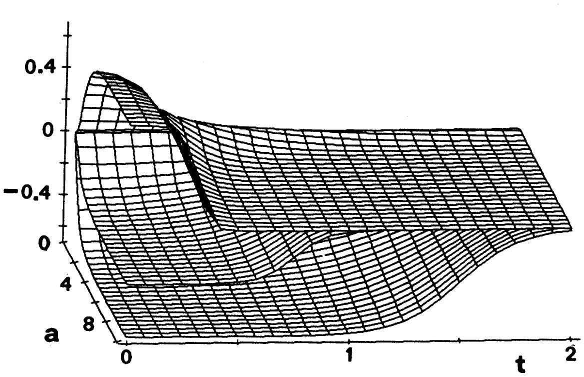

FIG. 1. A single soliton as the solution of a functional (difference-difference) KdV equation for $\delta = 0.6$, 1.5, 3 (from top to bottom), and $a = i\pi\kappa$, κ imaginary; see [2] for the exact formula.

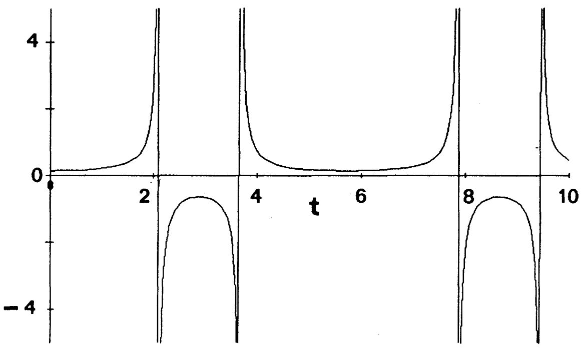

FIG. 2. A singular periodic solution equivalent to a single soliton for $\delta = 2$ and $\kappa = 1/(2\pi)$ real.

The operation S-lim can now be defined as follows. Let the Riemannian matrix $B \in C^{g\times g}$ be decomposed into blocks $B^{ss} \in C^{s\times s}, B^{pp} \in C^{p\times p}, B^{ps} \in C^{p\times s}$, and $B^{sp} = (B^{ps})^t$ where superscript t denotes transposed matrix. Of course $s + p = g$ and matrices B^{ss} and B^{pp} are also Riemannian. Let the argument $z \in C^g$ be similarly decomposed into $z_s \in C^s$ and $z_p \in C^p$.

The matrix $D^{ss} := \mathrm{Im}[\mathrm{diag}(B^{ss})]$ has s diagonal elements d_i. We define the S-lim operation as an s-fold limit

$$S\text{-lim}\ \Theta(z|B) : = \lim_{d_i\to\infty} \Theta\left[\begin{array}{c} z^s - iD^{ss}e^s/2 \\ z^p \end{array}\middle| \begin{array}{cc} B^{ss} & B^{sp} \\ B^{ps} & B^{pp}\end{array}\right] = : T(z), \qquad (39)$$

where all elements of diagonal matrix d_i tend to infinity and $e^s = (1,\ldots,1)^t \in Z^s$. It turns out that the limit for the fixed Riemannian matrix B exists always, and relation (1) in version (9) has the property (38) taking finally the form of

$$T(z+w)T(z-w) = \sum_{\epsilon} W_\epsilon(w)T(z+\epsilon/2)^2, \qquad (40)$$

where the sum is over g-dimensional hypercube $\epsilon \in Z_2^g$ and

$$T(z) = \sum_{n\in(Z_2)^s} \exp\left[i\pi(2\langle n, z^s\rangle + \langle n, \widetilde{B}^{ss}n\rangle)\right] \times\Theta(z^p + B^{ps}n|B^{pp}), \qquad (41)$$

with $\widetilde{B}^{ss} := B^{ss} - iD^{ss}$, i.e., having only real diagonal elements. The W function is now defined as

$$W_\epsilon(w) = 2^{-g} \sum_{\mu\in Z_2^g} (-1)^{\langle\epsilon,\mu\rangle} \exp(i2\pi\langle w,\epsilon\rangle) \times R_\epsilon(2w|2B)/R_\epsilon(0|2B), \qquad (42)$$

where

$$R_\epsilon(2w|2B) := \sum_{\mu^s} \exp\left\{-i\pi\langle\mu^s, [2w^s + \widetilde{B}^{ss}(\epsilon^s - \mu^s) + B^{sp}\epsilon^p]\rangle\right\} \times\Theta(2w^p + B^{pp}\epsilon^p + B^{ps}(\epsilon^s - 2\mu^s)|2B^{pp}), \qquad (43)$$

and in the last sum over μ^s, the summation is over such $\mu^s \in Z_2^s$ that also $(\epsilon^s - \mu^s) \in Z_2^s$. Of course ϵ^g is decomposed also in ϵ^s and ϵ^p parts.

It is clear that the formalism (39)–(43) leads to the processes in the form of multisolitons (s solitons) on the background of the p-phase quasiperiodic wave train. In the marginal situations when $s = 0$ or $p = 0$, one has pure quasiperiodic or pure multisoliton processes, provided the relevant system of dispersion equations is fulfilled.

In application to the dd-KdV equation here we shall discuss only the simplest nontrivial case of a soliton on the background of a periodic wave train ($s = p = 1$).

Rewriting (34) as

$$\sum_{i=1}^{3} \gamma_i W_\epsilon({}^i w) = 0, \qquad \text{with } \gamma = (1/\delta, 1, C), \qquad (44)$$

where now ${}^i w \in C^2 (i = 1, 2, 3)$ and taking into account (42) for any $\epsilon \in Z^2$ [i.e., $\epsilon = (0,0),(1,0),(0,1),(1,1)$], we obtain the following dispersion relation:

$$\sum_{i=1}^{3} \gamma_i \exp(i2\pi\langle {}^i w, \epsilon\rangle)R_\epsilon(2\,{}^i w|B) = 0 \qquad \text{for any } \epsilon \in (Z_2)^2, \qquad (45)$$

i.e., for $\epsilon = (0,0),(1,0),(0,1),(1,1)$. Let us ascribe index 1 to the s part (soliton) and index 2 to the p part (periodic). Instead of (20), through (41), F is given now by

$$F(z) = T(z) = \Theta(z_2|B_{22}) + \exp[i\pi(2z_1 + \widetilde{B}_{11})] \times\Theta(z_2 + B_{12}|B_{22}), \qquad (46)$$

where $\widetilde{B}_{11}$ is real and without losing generality can be chosen as zero. It is seen that the solution asymptotically ($z_1 \to \pm i\infty$) has the form of a periodic process, but there exists a shift between the left and right hand side asymptotics and the off-diagonal element of the starting B matrix (i.e., B_{12}) represents its measure.

Let us consider now the dispersion equation (45). We shall write explicitly the full system of four equations in order to show a peculiar property of such a mixed solution: the "dominance" of the periodic subprocess (or periodic phase) over the soliton one. For simplicity we assume $\widetilde{B}_{11} = 0$.

For $\epsilon = (0,0),(1,0),(0,1),(1,1)$ the relevant equations are

$$\sum_{i=1}^{3} \gamma_i \exp(i2\pi\,{}^i w_2)\Theta(2\,{}^i w_2|2B_{22}) = 0, \qquad (47)$$

$$\sum_{i=1}^{3} \gamma_i \exp(i2\pi\, {}^i w_2) \Theta(2\, {}^i w_2 + B_{22}|2B_{22}) = 0, \tag{48}$$

$$\sum_{i=1}^{3} \gamma_i \exp(i2\pi\, {}^i w_2) \Big[\exp(i2\pi\, {}^i w_1) \Theta(2\, {}^i w_2 + B_{12}|2B_{22}) + \exp(-i2\pi\, {}^i w_1) \Theta(2\, {}^i w_2 - B_{12}|2B_{22}) \Big] = 0, \tag{49}$$

$$\sum_{i=1}^{3} \gamma_i \exp(i2\pi\, {}^i w_2) \Big[\exp(i2\pi\, {}^i w_1) \Theta(2\, {}^i w_2 + B_{12} + B_{22}|2B_{22}) + \exp(-i2\pi\, {}^i w_1) \Theta(2\, {}^i w_2 - B_{12} + B_{22}|2B_{22}) \Big] = 0, \tag{50}$$

respectively. The quantities to be determined are ω_1, κ_1 and ω_2, κ_2 involved by means of ${}^i w_1$ and ${}^i w_2$, respectively (i=1,2,3), due to (33), and perhaps C. Observe that ω_2 and κ_2, i.e., the quantities which determine the periodic subprocess, can be obtained from Eqs. (47) and (48). However, these equations are the same for the single periodic process considered alone, without the presence of the soliton one. On the other hand, the quantities ω_1 and κ_1 which relate to the soliton subprocess depend on the periodic subprocess via (49) and (50), but not vice versa. This means that for the solution in the form of a soliton on the periodic wave train background, the periodic subprocess is primary to some extent.

The above conclusion is exactly the same as in the case of partial differential equations, e.g., KdV, and can be justified by energetic consideration. This effect is to some extent opposed to a common practice in a perturbational approach, where, in the presence of solitons, the periodic subprocess is assumed to be a small perturbation usually and hence considered as secondary.

An example of a dd-KdV solution in the form of a soliton on the periodic wave train background as an exact analytical solution of (26) according to (27), (45), and (46) is presented in Fig. 3. The preceding discussion concerning the "dominance" of the periodic subprocess relates also to this case and the parameters of the periodic phase are determined only by (47) and (48).

One can wonder why the technique originated in fact from the Hirota approach is adaptable for PDE, ODE, and functional equations as well.

Two facts play a crucial role. The first one is the class of functions involved, which satisfy the addition relation. The second is the definition of a discrete version of the Hirota bilinear differential operator D [11], let us call it $\widetilde{D}_a$, which in the limit tends to D, and together with the addition property (1) admits the summation of $\widetilde{D}$ operators with different powers. Let us define the operator $\widetilde{D}_a$ for functions f and g of a scalar argument and a step a being a parameter through the relation

$$\widetilde{D}_a^n(f,g) := \sum_{k=0}^{n} \frac{(-1)^k n!}{k!(n-k)!a^n} f[x + (n/2 - k)a] \times g[x - (n/2 - k)a]. \tag{51}$$

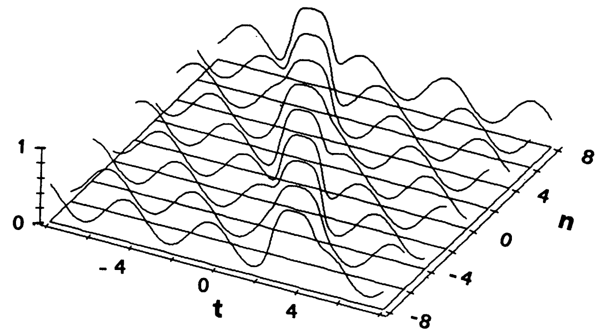

FIG. 3. Solution of the functional (difference-difference) KdV functional equation in the form of the a periodic wave train background. Parameters of the solution $\delta = 2$, $B_{22} = i1.44$, $B_{12} = -i0.5$; soliton: $\pi\kappa_1 = -i1.656$, $\pi\omega_1 = i3.277$; wave train: $\pi\kappa_2 = i1$, $\pi\omega_2 = -i1.108$; see Eqs. (47)–(50).

The definition of $\widetilde{D}_a$ is of course not unique. The definition proposed here differs from the definition of a discrete operator reported, e.g., in [12].

It is obvious that the limit

$$\lim_{a\to 0} \widetilde{D}_a^n(f,g) = D^n(f,g) = \frac{\partial^n}{\partial a^n} f(x+a)\, g(x-a) \Big|_{a=0} \tag{52}$$

coincides with the definition of the bilinear differential Hirota operator [11]. The second property is also fundamental. If the functions f have the addition property (1), all operators $\widetilde{D}_a^n(f,f)$ have the representation

$$\widetilde{D}_a^n(f,f) := \sum_{k=0}^{n} \frac{(-1)^k n!}{k!(n-k)!a^n} f[x + (n/2 - k)a] \times f[x - (n/2 - k)a] = \sum_{\epsilon} Y_\epsilon(n) Z_\epsilon(x), \tag{53}$$

where

$$Y_\epsilon(n) = \sum_{k=0}^{n} \frac{(-1)^k n!}{k!(n-k)!a^n} W_\epsilon((n/2 - k)a). \tag{54}$$

This means that $\widetilde{D}_a^n(f,f)$ of different n can be easily added, since the "basis" does not depend on the order of the operator. As a final result we obtain the dispersion equations.

The last conclusion is identical to the case of differential operators and therefore the author is convinced that the addition property (1) plays a fundamental role for PDE, ODE, and functional equations of a soliton type.

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